SYMMETRY OF GRAPH

A Project Report
submitted by

Trupthi S (2120317)
Varada M T (2121412)
Sreevidya K M (2121411)
Adithya K K (2121401)
Roshel Lisa D'sa (2120313)

Under the guidance of
Ms. Priya Monteiro
to

LUCET ET ARDET
ESTD : 1880

ST ALOYSIUS

(DEEMED TO BE UNIVERSITY)

In partial fulfillment of the requirements for Continuous Internal Assessment under NEP

Bachelor of Science
VI Semester

Department of Mathematics

February, 2024

CERTIFICATE

This is to certify that the project report entitled "Symmetry of graph" is authentic work carried out by **Varada M T (2121412), Trupthi S (2120317), Sreevidya K M (2121411), Adithya K K (2121401), Roshel Lisa D'sa (2120313)** under the guidance of Ms. Priya Monteiro from the Department of Mathematics, St Aloysius (Deemed to be University), Mangaluru.

The same is being submitted to the UG Department of Mathematics, St Aloysius (Deemed to be University) in partial fulfillment of the requirements for the **Continuous Internal Assessment of VI Semester, Bachelor of Science**. No part of this thesis has been presented for the award of any other degree.

Signature of the guide, HOD

Ms Priya Monteiro

Date of submission: 21/02/2024

Contents

Graph theory

maths5874

23rd February 2024

1 Introduction

This introduction to graph theory focuses on well-established topics, covering primary techniques and including both algorithmic and theoretical problems. The algorithms are presented with a minimum of advanced data structures and programming details. This thoroughly corrected 1988 edition provides insights to computer scientists as well as advanced undergraduates and graduate students of topology, algebra, and matrix theory. Fundamental concepts and notation and elementary properties and operations are the first subjects, followed by examinations of paths and searching, trees, and networks. Subsequent chapters explore cycles and circuits, planarity, matchings, and independence. The text concludes with considerations of special topics and applications and extremal theory. Exercises appear throughout the text.

Graph theory is now used today in the physical sciences, social sciences, computer science, and other areas. *Introductory Graph Theory* presents a non-technical introduction to this exciting field in a clear, lively, and informative style.

Author Gary Chartrand covers the important elementary topics of graph theory and its applications. In addition, he presents a large variety of proofs designed to strengthen mathematical techniques and offers challenging opportunities to have fun with mathematics.

Ten major topics – profusely illustrated – include Mathematical Models, Elementary Concepts of Graph Theory, Transportation Problems, Connection Problems, Party Problems, Digraphs and Mathematical Models, Games and Puzzles, Graphs and Social Psychology, Planar Graphs and Coloring Problems, and Graphs and Other Mathematics.

A useful Appendix covers Sets, Relations, Functions, and Proofs, and a section devoted to exercises – with answers, hints, and solutions – is especially for anyone encountering graph theory for the first time.

Undergraduate mathematics students at every level, puzzles, and mathematical hobbyists will find well-organized coverage of the fundamentals of graph theory in this highly readable and thoroughly enjoyable book.

2 Automorphism

2.1 Definition

A group automorphism is an isomorphism from a group to itself. If G is a finite multiplicative group, an automorphism of G can be described as a way of rewriting its multiplication table without altering its pattern of repeated elements. For example, the multiplication table of the group of 4th roots of unity $G = \{1, -1, i, -i\}$ can be written as shown above, which means that the map defined by

	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

is an automorphism of G.

In general, the automorphism group of an algebraic object O, like a ring or field, is the set of isomorphisms of that object O and is denoted $\text{Aut}(O)$. It forms a group by composition of maps. For a fixed group G, the collection of group automorphisms is the automorphism group $\text{Aut}(G)$.

2.2 Properties of automorphismm graphs

An automorphism of a graph is a graph isomorphism with itself, i.e., a mapping from the vertices of the given graph G back to vertices of G such that the resulting graph is isomorphic with G. The set of automorphisms defines a permutation group known as the graph's automorphism group. For every group Γ, there exists a graph whose automorphism group is isomorphic to Γ. The automorphism groups of a graph characterize its symmetries and are therefore very useful in determining certain of its properties.

The group of graph automorphisms of a graph G may be computed in the Wolfram Language using `GraphAutomorphismGroup[g]`, the elements of which may then be extracted using `GroupElements`. A number of software implementations exist for computing graph automorphisms, including nauty by Brendan McKay and SAUCY2, the latter of which performs several orders of magnitude faster than other implementations based on empirical tests.

Precomputed automorphisms for many named graphs can be obtained using `GraphData[graph, "Automorphisms"]`, and the number of automorphisms using `GraphData[graph, "Automorphism Count"]`.

Graph Automorphism Star:

Similarly, the star graph S_4 has six automorphisms: $(1, 2, 3, 4)$, $(1, 3, 2, 4)$, $(2, 1, 3, 4)$, $(2, 3, 1, 4)$, $(3, 1, 2, 4)$, $(3, 2, 1, 4)$, illustrated above. More generally, as is clear from its symmetry: $|\text{Aut}(S_n)| = (n - 1)!$ for $n \geq 3$. Precomputed automorphisms for many named graphs can be obtained using `GraphData[graph, "Automorphisms"]`, and the number of automorphisms using `GraphData[graph, "AutomorphismCount"]`.

Graph Automorphism Grid Graph:

For example, the grid graph $G_{2,3}$ has four automorphisms: $(1,2,3,4,5,6)$, $(2,1,4,3,6,5)$, $(5,6,3,4,1,2)$, and $(6,5,4,3,2,1)$. These correspond to the graph itself, the graph flipped left-to-right, the graph flipped up-down, and the graph flipped left-to-right and up-down, respectively, illustrated above. More generally, as is clear from its symmetry:

$$|\mathrm{Aut}(G_{m,n})| = \begin{cases} 1 & \text{for } m = n = 1; \\ 2 & \text{for } m = 1 \text{ or } n = 1; \\ 4 & \text{for } m \neq n \text{ and } m, n > 1; \\ 8 & \text{for } m = n > 1. \end{cases}$$

In our first example, the set V_G would be the set of cities, and $\{u_i, u_j\} \in E_G$ if the cities represented by u_i and u_j were connected by a road. In the second example, V_G would be the set of people and $\{u_i, u_j\} \in E_G$ if the people represented by u_i and u_j were friends (this is, of course, assuming that friendship is a symmetric relation).

While these examples are very similar, there are some important potential differences. Suppose that we are building a graph to represent cities and the roads between them because we want to know the shortest route to drive between any two cities. Then in this case it would be important to know how long each of the roads is. We could then assign a weight to any given edge of our graph. Alternatively, suppose we want to know how many ways there are to get from one city to another. Then it would be important to know if there was more than one direct road between two cities, or even if there was a scenic loop that went from one city back to itself. To faithfully represent this in our graph, we would need to allow there to be multiple edges between vertices in addition to edges between a vertex and itself. In contrast, we consider the case of a group of people and their friendships. It clearly does not make sense to assign a quantifier to the friendship between any two people, nor does it make sense to have two people be friends twice over or someone be friends with themselves. Therefore in this case we would like to only allow our graph to have edges between two distinct vertices, and no more than one edge per vertex pair. Such a graph is said to be simple.

In this paper, we will only be considering simple graphs. For more information on other kinds of graphs, see Bogart [2] or Bona [3].

One useful concept that helps us discuss graphs is as follows: the degree of a vertex is equal to the number of edges incident at that vertex. Therefore, to find the degree of a given vertex $v \in V_G$, we must simply count the number of edges $e \in E_G$ that have v as one of their two entries. Knowing the degrees of vertices present in the graph will help to compare graphs to each other.

Just as with groups, there are a few graphs that are common enough to have their own standard notation. The three that we will be using in this paper are: The Complete Graph on n vertices K_n in which every vertex is connected to all $n-1$ other vertices. The Path Graph on n

vertices P_n consists of n vertices $v_0, v_1, v_2, \ldots, v_{n-1}$ and $n-2$ edges, where $\{v_i, v_{i+1}\} = e_i \in E_{P_n}$ for all $0 \leq i \leq n-2$.

The Cycle Graph C_n consists of the path graph P_n, but with the extra edge $\{v_{n-1}, v_0\}$

It is only natural to want to use some of our tools from our study of Abstract Algebra to understand these graphs better. We start by refining the idea of a group automorphism defined in Definition. **Definition 2.2.** A graph automorphism of G is a permutation φ on the set of vertices V_G that satisfies the property that $\{u_i, u_j\} \in E_G$ if and only if $\{\varphi(u_i), \varphi(u_j)\} \in E_G$.

Now that we have a definition of a graph automorphism that parallels Definition 1.5, it is not a stretch to wonder if the set of all graph automorphisms on a particular graph G forms a group, as was found with group automorphisms in Theorem 1.6. This is in fact the case.

Theorem 2.3. The set $\mathrm{Aut}(G)$ of all graph automorphisms of a graph G forms a group under function composition.

The proof of this fact follows that of Theorem 1.6 almost precisely. We also note that a graph and its complement are very similar in structure. This leads us to believe that there might be some sort of relationship between their automorphism groups. This turns out to be the case, as outlined in the next theorem.

Theorem 2.4. Given any graph G, $\mathrm{Aut}(G) = \mathrm{Aut}(G^{)}$.

Proof. We will proceed by showing set inclusion in both directions.

Direction 1: Let $\sigma \in \mathrm{Aut}(G)$. Suppose $e \notin E_G$ is an edge. By the definition of the complement of a graph, $e \in E_G$. Since σ is a graph automorphism, $\sigma(e) \notin E_G$, implying $\sigma(e) \in E_G$. Thus, σ is also an automorphism of G, and hence $\sigma \in \mathrm{Aut}(G^{)}$.

Direction 2: Let $\tau \in \mathrm{Aut}(G^{)}$. Since G is isomorphic to G, we can interchange G and G. Thus, if $\tau(e) \notin E_G$ for some edge $e \notin E_G$, then $\tau(e) \notin E_G$, implying $\tau(e) \in E_G$. Hence, $\tau \in \mathrm{Aut}(G)$.

Therefore, we have shown that $\mathrm{Aut}(G) = \mathrm{Aut}(G^{)}$.

3 Edge Transitivity

3.1 Definition

In the mathematical field of graph theory, an edge-transitive graph is a graph G such that, given any two edges ePrecomputed automorphisms for many named graphs can be obtained using `GraphData[graph, "Automorphisms"]`, and the number of automorphisms using `GraphData[graph, "AutomorphismCount"]`.

3.2 Properties of Edge transitive graphs

- Symmetry: Edge transitive graphs exhibit a high degree of symmetry, as any pair of edges can be transformed into one another by a graph automorphism.
- Regularity: Edge is especially often regular, meaning that each vertex has the same number of incident edges. However, not all edge-transitive graphs are regular.
- Vertex transitivity:While edge transitivity does not necessarily imply vertex transitivity, many edge-transitive graphs are also vertex-transitive, meaning that for any two vertices v1 and v2, there exists an automorphism mapping v1 to v2.
- Cayley Graphs: Many edge transitive graphs are Cayley graphs, which are graphs that represent the st, structure of a group.
- Classification: Edge-transitive graphs are classified based on their underlying structures, such as circulants, symmetric graphs and other algebraic combinatorial structures.

3.3 Automorphism

An automorphism of a graph is a graph isomorphism with itself, i.e., a mapping from the vertices of the given graph G back to vertices of G such that the resulting graph is isomorphic with G. The set of automorphisms defines a permutation group known as the graph's automorphism group. An edge-transitive graph is an undirected graph in which every edge may be mapped by automorphisms to any other edge.

3.4 Classification of edge transitive graphs

- Circulant graph:A circulant graph is formed in a systematic way based on a set of integers. Its edges are arranged in a circular or cyclic manner, and the graph exhibits a regular pattern.
- Symmetric graph: a graph is symmetric if its automorphism group acts transitively on ordered pairs of adjacent vertices (that is, upon edges considered as having a direction).
- Regular Edge-Transitive Graphs:These are graphs where every vertex has the same number of connections automorphisand adjacent to any edge with any other edge while maintaining the graph's overall structure.
- Complete Graphs:In a complete graph, every pair of vertices is directly connected by an edge. You can switch any edge with any other edge in the graph.
- Prism Graphs:These graphs are like stacking two complete graphs on top of each other. They maintain edge-transitivity, and you can swap edges in a certain way.

3.5 Applications of Edge Transitive Graphs

- Network Design:In the design of communication networks, transportation networks, and social networks, edge-transitive graphs can be used to model and understand the symmetric distribution of connections. This can lead to more efficient and balanced network designs.
- Error-Correcting Codes:Edge-transitive graphs are employed in coding theory for constructing error-correcting codes. The symmetry properties of these graphs can be exploited to design codes that can correct errors in transmitted data.
- Group Theory and Algebraic Structures:Edge-transitive graphs are closely related to group theory. They can be used to construct Cayley graphs, which are graphs associated with groups. Understanding edge-transitive graphs contributes to the study of algebraic structures and their applications.
- Random Walks and Markov Chains:The symmetry properties of edge-transitive graphs make them useful in the analysis of random walks and Markov chains. These graphs often provide a structured environment for studying the properties of random processes.
- Parallel Computing:In parallel computing, edge-transitive graphs can be used to model and design interconnection networks. The symmetry of these graphs can lead to balanced communication patterns among processors, contributing to efficient parallel processing.

- Graph Isomorphism Testing:Edge-transitive graphs are useful in the study of graph isomorphism, which involves determining whether two graphs are essentially the same. The symmetry properties of edge-transitive graphs can simplify certain isomorphism testing procedures.
- Coding and Cryptography:The study of edge-transitive graphs plays a role in coding theory and cryptography. Certain cryptographic protocols and algorithms may benefit from the unique symmetries and properties exhibited by edge-transitive graphs.
- Combinatorial Designs:Edge-transitive graphs are often involved in the construction of combinatorial designs such as block designs and orthogonal arrays, which have applications in experimental design and coding theory.

3.6 Open problems in edge transitivity

- Classification and Enumeration:Achieving a comprehensive classification of all edge-transitive graphs and enumerating them systematically remains a challenging problem. While certain families are well-understood, a complete classification for all possible cases is an open question.
- Automorphism Groups:Understanding the structure and properties of the automorphism groups of edge-transitive graphs is a challenging problem. Characterizing the automorphism groups in a more general and systematic way is an ongoing research direction.
- Connection with Other Graph Properties:Investigating the relationships between edge-transitivity and other graph properties, such as chromatic number, clique number, or girth, is an open problem. Understanding how edge-transitivity interacts with other graph-theoretic concepts is an ongoing research area.
- Random Graph Models:Studying the behavior of edge-transitive graphs within various random graph models is an open problem. Understanding the emergence of edge-transitivity in random graph processes remains an area of exploration.
- Applications in Real-World Networks:Investigating the applicability of edge-transitive graphs in modeling and analyzing real-world networks is an open research direction.

4 vertex Transitivity

4.1 Definition

Vertex transitivity is a concept in graph theory that describes a property of graphs. A graph is said to be vertex-transitive if, for every pair of vertices in the graph, there exists an automorphism of the graph that maps one vertex to the other. In simpler terms, a graph is vertex-transitive if there is a symmetry in the arrangement of vertices such that you can transform any vertex into any other vertex by applying an automorphism.

4.2 Automorphism

An automorphism of a graph is a bijective (one-to-one and onto) function from the set of vertices to itself, such that the edges are preserved. In other words, if there is an edge between vertices A and B, after the automorphism, there should still be an edge between the images of A and B.

4.3 Example:

- Consider a graph where every vertex has the same set of neighbors. If there is an automorphism that maps any vertex to any other vertex, then the graph is vertex-transitive.

4.4 Applications:

Vertex transitivity is a useful concept in various applications, such as network analysis, chemistry (molecular graphs), and computer science (circuit design). Understanding the symmetr y properties of a graph can provide insights into its structure and behavior.

1. **Network Routing:** In communication networks, particularly in routing algorithms, edge-transitive graphs can be advantageous. The symmetric nature of the edges can simplify routing decisions and optimize the flow of information.
2. **Circuit Design:** Edge-transitive graphs can be relevant in the design of circuits, where the symmetry of connections between components might be advantageous for certain applications.
3. **Chemical Graph Theory**: In chemistry, molecules can be represented as graphs, where atoms are vertices and chemical bonds are edges. Edge transitivity in these molecular graphs can provide insights into the structural symmetry of molecules.
4. **Social Network Analysis:** In social networks, where edges represent relationships between individuals, edge transitivity might indicate similar

relationship patterns between different pairs of individuals. Understanding this symmetry can help in analyzing the structure of social connections.

5. **Image Processing:** Edge-transitive graphs can be applied in image processing, where edges represent image features. The symmetry in the edge structure might be useful for certain image recognition or pattern analysis tasks.

6. **Transportation Networks:** Edge transitivity can be considered in transportation networks, such as road or rail systems. Symmetric connectivity between different segments of the network might influence the efficiency of transportation routes.

7. **Error Detection and Correction:** In error-detection codes and communication protocols, edge-transitive graphs can be employed for efficient error correction. The symmetry in edge connections may aid in identifying and correcting errors in the communication process.

8. **Algorithm Design:** When designing algorithms for graphs, considering edge transitivity can lead to more efficient solutions. Symmetric properties of edges might be exploited to optimize certain graph-based algorithms.

9. **Wireless Sensor Networks:** In the context of wireless sensor networks, where edges represent communication links between sensors, edge transitivity can impact the reliability and efficiency of data transmission.

10. **Game Theory:** In certain game-theoretic models represented as graphs, edge transitivity might have implications for strategies and equilibrium points in the game.

4.5 Formal Definition:

A graph G is vertex-transitive if, for every pair of vertices u and v in G, there exists an automorphism ϕ of G such that $\phi(u) = v$

4.6 Study and Implement Algorithms:

Research and understand algorithms for determining vertex transitivity in graphs.

Compare the performance of different algorithms on various types of graphs.

4.7 Graph Visualization:

- Develop a tool to visualize and analyze vertex transitivity in graphs.
- Allow users to input or generate graphs and visually explore their vertex-transitive properties.
- Explore real-world applications of vertex transitivity in fields such as social networks, transportation systems, or biological networks.

- Investigate how vertex transitivity can be used to model and analyze these systems.

4.8 Random Graph Generation:

- Create a program that generates random graphs with a specified level of vertex transitivity.
- Explore the properties of these graphs and how they differ from non-transitive graphs.

4.9 Network Robustness:

- Investigate the impact of vertex transitivity on the robustness of networks.
- Simulate attacks or failures in a network and analyze how the vertex transitivity affects the network's ability to withstand such disruptions.

4.10 Educational Tools:

- Study how vertex transitivity relates to community structure in networks.
- Develop algorithms to identify and analyze communities in vertex-transitive graphs.
- Create educational materials or interactive tools to help others understand the concept of vertex transitivity and its significance in graph theory.

4.11 Challenges and Limitations :

- **Algorithmic Challenges:-** Developing efficient algorithms to handle vertex transitivity can be complex. The specific structure of the graph and the chosen algorithm can influence the accuracy and efficiency of the calculations.

5 Regular graph

5.1 Introduction

Graph theory is a branch of mathematics that explores the relationships between entities through interconnected nodes and edges. Regular graphs are a fundamental concept within graph theory, playing a crucial role in various applications such as computer science, communication networks, and social sciences. In this document, we delve into the intricacies of regular graphs, examining their definitions, properties, and real-world applications.

5.2 Definition

A regular graph is a type of undirected graph in which each vertex has the same degree, i.e., the same number of edges incident to it. The degree of a vertex is the count of edges connected to it. Therefore, in a regular graph, all vertices have identical degrees, giving the graph a certain symmetry. Regular graphs are classified based on the degree of their vertices.

- **k-Regular Graphs** In a k-regular graph, each vertex has a degree of exactly k. These graphs are particularly interesting because they exhibit a high degree of symmetry. They are further classified into simple and multigraphs based on whether multiple edges between the same pair of vertices are allowed or not.

5.3 Properties

1. **Uniform Degree** The most defining property of regular graphs is that all vertices have the same degree. In a k-regular graph, each vertex has exactly k edges incident to it. This uniformity in degrees creates a balanced structure.

2. **Symmetry** Regular graphs exhibit a high degree of symmetry. The uniform distribution of edges among vertices results in a graph that looks similar from various perspectives. This symmetry can be visually appealing and is often utilized in design and network planning.

3. **Edge Coloring** Regular graphs have interesting properties related to edge coloring. For example, the chromatic index of a k-regular graph is known to be k when k is even, and k or k+1 when k is odd.

4. **Cycle Structure** Regular graphs often have a cyclic structure. This is particularly true for regular graphs with an even degree, where cycles play a significant role in the overall connectivity of the graph. The cycles contribute to the cohesive nature of the graph.

5.4 Applications

1. **Network Topology** Communication Networks: Regular graphs are frequently used to model communication networks, such as computer networks, where each node represents a device (like a computer or router), and edges represent communication links. The regularity ensures a uniform distribution of connections, optimizing communication and reducing bottlenecks.

2. **Coding Theory** Error-Correcting Codes: Regular graphs are employed in coding theory to design error-correcting codes. The vertices represent codewords, and edges represent potential errors that can be corrected. The regular structure allows for efficient error detection and correction algorithms, vital in reliable data transmission.

3. **Chemistry** Molecular Structure: In chemistry, regular graphs are used to model molecular structures. Atoms are represented by vertices, and edges correspond to chemical bonds. The regularity of such graphs reflects the consistent valency of atoms, providing insights into the stability and reactivity of molecules.

4. **Robotics** Sensor Networks: In robotics, regular graphs are employed in the design of sensor networks. The regularity ensures that each sensor node has a similar communication load and coverage, contributing to a more robust and evenly distributed sensing infrastructure.

5. **Computer Science** Parallel Computing: Regular graphs are used in parallel computing to model the communication and coordination between processors. The uniform degree distribution helps in designing efficient parallel algorithms.

5.5 Graph Theory Concepts

- **Adjacency Matrix** The adjacency matrix of a regular graph is a square matrix that represents the connections between vertices. In a regular graph, the adjacency matrix is often characterized by a constant value for each row (or column), reflecting the regularity of the graph.

- **Incidence Matrix** The incidence matrix is another representation of a graph that relates vertices to edges. In regular graphs, the incidence matrix reflects the constant degree of each vertex.

- **Cycles and Paths** Regular graphs may exhibit cycles, which are closed paths in the graph. Understanding the presence and properties of cycles is important in analyzing the structure of regular graphs.

- **Eigenvalues and Eigenvectors** The eigenvalues and eigenvectors of the adjacency matrix or Laplacian matrix of a regular graph provide important information about its structure. These concepts are used in spectral graph theory to study regular graphs.

- **Isomorphism** Regular graphs may be isomorphic, meaning that they have the same structure but may have different vertex labels. Understanding graph isomorphism can be crucial when comparing and analyzing regular graphs.
- **Automorphisms** Regular graphs often have a high degree of symmetry. The study of automorphisms, which are isomorphisms from a graph to itself, can provide insights into the symmetries present in regular graphs.

5.6 Challenges and Open Problems

- Existence of Hamiltonian Cycles
- Spectral Gap Conjecture
- Hamiltonian Decompositions
- Graph Reconstruction
- Computational Complexity

5.7 Examples

- Complete Graphs
- Cycle Graphs
- cube Graph
- Regular Bipartite Graphs
- Kautz Graphs
- Petersen Graph
- Dodecahedral Graph

6 Cayley graph

6.1 Introduction

In mathematics, a Cayley graph, also known as a Cayley color graph, Cayley diagram, group diagram, or color group, is a graph that encodes the abstract structure of a group. Its definition is suggested by Cayley s theorem (named after Arthur Cayley) and uses a specified set of generators for the group. It is a central tool in combinatorial and geometric group theory. The structure and symmetry of Cayley graphs make them particularly good candidates for constructing expander graphs. The only groups that can give planar Cayley graphs are exactly $Z_n, Z_2 \times Z_n, D_n, S_4, A_4$, and A_5, as proved by Maschke (1896). The following table lists some graphs that are undirected versions of Cayley graphs generated by small numbers of small permutations. The group G acts on itself by left multiplication (see Cayley's theorem). This may be viewed as the action of G on its Cayley graph. Explicitly, an element $h \in G$ maps a vertex $g \in V(\Gamma)$ to the vertex $hg \in V(\Gamma)$. The set of edges of the Cayley graph and their color is preserved by this action: the edge (g, gs) is mapped to the edge (h, hgs), both having color c_s. In fact, all automorphisms of the colored directed graph Γ are of this form, so that G is isomorphic to the symmetry group of Γ.

6.2 definition 1.1

We begin by giving a very brief introduction to the topic of graphs with an emphasis on Cayley graphs, which will be the focus of all of our examples in this section. We assume some familiarity with groups.

A graph is a pair $G = (V, E)$ where V is a set of points called vertices and E is a collection of vertex pairs called edges. A loop is an edge whose associated vertices are the same.Graph contains one loop.

Graphs can also have an explicit orientation to their edges. A graph is directed if its edges consist of ordered (rather than unordered) pairs of vertices.

One very important aspect of graphs is that they can depict relations between the elements of a group.

6.3 definition 1.2

Let G be a group and let $S \subset G$ be a set of generators. A Cayley graph is a graph where the following hold:

1. $V = G$ and

2. given any two vertices $v_1, v_2 \in V$, there is an edge from v_1 to v_2 if and only if $v_1 s = v_2$ for some $s \in S$.

In other words, the vertices of a Cayley graph are group elements, and the edges between them represent multiplication by group generators. The Cayley graph of a group is not necessarily unique but depends on the choice of the generating set. If a group has multiple generating sets, we will specify the one we are using.

A space is simply connected if it is path-connected and if it has a trivial fundamental group. We propose that our simply connected covering space is $X = \{[\gamma]\}$, where γ is a path in X with starting point x_0. Take $p : X \to X$ to be the function that maps $[\gamma]$ to $\gamma(1)$, the fixed endpoint of $[\gamma]$ in X. By path-connectedness of X, we can take any point of X to be $\gamma(1)$. This implies that p is surjective. Let U be the topology on X consisting of the path-connected open sets U whose fundamental groups map trivially to the fundamental groups of X.

6.4 cayley complex

The Cayley complex X of a group X is the Cayley graph of X that has a 2-cell attached by its boundary to each loop at each vertex.

We use the notation X in this definition because the Cayley complex is a simply connected covering space, and is thus the universal cover of X. We know from our previous discussion that if X is the universal cover of X, then the deck transformation group $G(X)$ is isomorphic to $\pi_1(X)$.Moreover, if H is a subgroup of $\pi_1(X)$. corresponding to some covering space $X_H = X/H$, then $\pi_1(X/H)$ is isomorphic to H. With this, we can now look at a couple of concrete examples.

(1) X = Z/3Z

The graph of X in Figure 5 has three vertices and one loop that can be based at each of them. Thus, we see that the Cayley complex X˜ is three disks (two-cells) glued to each other on top of the graph. The deck transformation group of X˜ is Z/3Z. It acts on the disks in X˜ by rotations $\frac{2\pi n}{3}$.

(2) X = D4

Consider the subgroup of D4 generated by 90 degree rotations of the square, or 1, x, x2
, x3. This subgroup is isomorphic to Z/4Z, and we will refer to it as such. Looking at the graph of X in Figure 5, quotienting X˜ by Z/4Z shrinks down the inner and outer squares of the graph whose sides correspond to multiplications by x and deletes the three "petals" corresponding to multiplications by y that are not attached to e. Thus, X/˜ Z/4Z is the above graph Z/2Z with four 2-cells attached. This is the covering space of Z/4Z . The fundamental group of this new complex is exactly Z/4Z.

6.5 Cayley Graphs as Topological Spaces

We are ready to perform some computations. We will take the Cayley graphs and apply the topological theory we developed through the construction of Cayley complexes. Our emphasis here will not be on rigor but on gaining insight into the algebraic properties of the groups encoded by these Cayley graphs. We begin with some new terminology.

6.6 definition 1.1

An n-dimensional open cell, or n-cell, is a topological space homeomorphic to the n-dimensional open ball. For our purposes, all that is necessary to know about n-cells is that a 0-cell is a point, a 1-cell is a line segment, and a 2-cell is a polygon. We can "glue" together these cells to build a new topological space.

6.6.1 Deck Transformations

Building on the covering space theory we have been developing, we now present a final technique for computing fundamental groups of spaces. Once again, we assume some knowledge of groups.

Deck transformations can be thought of as "shufflings" of different covering spaces $\tilde{X}$, where the spaces are analogous to cards in a deck. This general idea is depicted in Figure 5. One can check that the set of deck transformations of a covering space under composition is a group, which we will denote by $G(\tilde{X})$. We have actually already dealt with deck transformations in the previous section, when we defined what it meant for two covering spaces to be isomorphic. Theorem 1 Let $p, X, \tilde{X}$, and H be defined as in Proposition 2.17. If $\tilde{X}$ is a regular covering space of (X, x_0), then $G(\tilde{X})$ is isomorphic to $\pi_1(X, x_0)/H$. It follows immediately that if $\tilde{X}$ is the universal cover of X, then $G(\tilde{X})$ is isomorphic to $\pi_1(X)$. As G is a group, we can discuss the group actions of G on $\tilde{X}$.

Let Y be a space acted on by a group G. Actions of G are called covering space actions if each $y \in Y$ is contained by a neighborhood U for which $g_1(U) \cap g_2(U) \neq \emptyset$ implies $g_1 = g_2$.

Let G be a group whose elements are covering space actions on a space Y. If Y is path-connected and locally path-connected, then G is isomorphic to $\pi_1(Y/G)/(p^*(\pi_1(Y)))$.

A covering space $p : \tilde{X} \to X$ is regular if for each $x \in X$ and each pair of lifts $\tilde{x}, \tilde{x}_0$ of x, there is a deck transformation $\tau : \tilde{X} \to \tilde{X}$ such that $\tau(\tilde{x}) = \tilde{x}_0$.

In other words, the lifts of an evenly covered point to a regular covering space differ by deck transformations. This next proposition gives another way of precisely identifying regular covering spaces.Therefore its proved.

7 Conclusion

In conclusion, the symmetry of a graph can provide valuable insights into its structure and behavior. Symmetrical graphs often exhibit certain patterns and properties that can aid in analysis and problem-solving. Identifying symmetries can help simplify complex graphs and reveal underlying relationships between vertices and edges. Additionally, symmetrical graphs are often visually appealing and can be easier to interpret. However, not all graphs possess symmetrical properties, and the presence or absence of symmetry depends on various factors such as the graph's topology, connectivity, and degree distribution. Overall, understanding symmetry in graphs is essential for both theoretical studies and practical applications in fields such as mathematics, computer science, and network analysis.

By identifying and exploiting symmetry, mathematicians and scientists can simplify complex problems and uncover underlying patterns and relationships within graphs. Additionally, symmetry plays a crucial role in the aesthetic appeal of graphs and other visual representations, enhancing their clarity and comprehensibility. Overall, the study of symmetry in graphs enriches our understanding of mathematical concepts and their applications across various disciplines.

8 Reference

- https://en.wikipedia.org/wiki/Automorphism
- https://www.overleaf.com/learn/latex/Tutorials
- https://mathworld.wolfram.com/Edge-TransitiveGraph.html
- https://en.wikipedia.org/wiki/Vertex-transitive-graph
- https://www.geeksforgeeks.org/regular-graph-in-graph-theory
- https://en.wikipedia.org/wiki/Cayley-graph
- https://mathworld.wolfram.com/GraphAutomorphism.html