A Study On Positive Definite and Semidefinite Matrices

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Outline

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- The German mathematician Carl Friedrich Gauss made significant contributions to the study of positive definite forms.
- Positive definite and positive semidefinite matrices became foundational concepts in modern mathematics, with applications spanning various disciplines, including computer science, machine learning, and statistics.

Basic Definitions

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An Eigenvector of $n \times n$ matrix A is a nonzero vector x such that $Ax = \lambda x$ for some scalar λ. A scalar λ is called an Eigenvector value of A if there is nontrivial solution x of $Ax = \lambda x$ such that x is called an Eigenvector corresponding to λ.

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The spectrum of $A \in \mathrm{M}_n$ is the set of all $\lambda \in \mathbb{C}$ that are Eigenvalues of A, we denote this set by $\sigma(A)$.

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A Hermitian matrix $A \in M_n$ is said to be *positive definite* if

$$x^*Ax > 0 \quad \textit{for all nonzero } x \in \mathbb{C}^n$$

it is *positive semidefinite* if

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Example

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Where A is positive semidefinite but not positive definite.

Theorem

Each eigenvalue of a positive definite (respectively, positive semidefinite) matrix is a positive (respectively, nonnegative) real number.

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Let A be a positive definite matrix for any $x \neq 0$. Suppose $Av = \lambda v$.Then we get, $v^T Av = \lambda v^T v$. Since A is positive definite, $v^T Av > 0$.

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Let A be a positive definite matrix for any $x \neq 0$. Suppose $Av = \lambda v$.Then we get, $v^T Av = \lambda v^T v$. Since A is positive definite, $v^T Av > 0$. Also, since $v^T v = \|v\|^2 > 0$, $v \neq 0$, it follows that $\lambda = \frac{v^T Av}{v^T v}$ is positive. Hence, the proof. □

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Corollary

A Hermitian matrix A is positive definite if and only if it is **congruent to the identity.*

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If A is congruent to I, then there exists a nonsingular matrix Q such that $A = Q^*IQ = Q^*Q$. Since Q is nonsingular, it is invertible.

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Since $A = I$ which implies that A is positive definite. □

The Schur Product Theorem

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Definition

If $A = [a_{ij}] \in \mathbb{M}_{m,n}$ and $B = [b_{ij}] \in \mathbb{M}_{m,n}$, then the Hadamard product (Schur product) of A and B is the entrywise product matrix $A \circ B = [a_{ij}b_{ij}] \in \mathbb{M}_{m,n}$.

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Example

Let A and B two positive semidefinite matrices in $\mathbb{R}^{2\times 2}$.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$$

Then,

$$A \circ B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \circ \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 15 \end{bmatrix}$$

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So $a_{ij}x_i x_j$ and $b_{ij}x_i x_j$ are both non-negative.

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So $a_{ij}x_ix_j$ and $b_{ij}x_ix_j$ are both non-negative. Therefore, their product $a_{ij}b_{ij}x_ix_j$ is also non-negative. Thus, $x^T(A \circ B)x$ is non-negative.

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So $a_{ij}x_ix_j$ and $b_{ij}x_ix_j$ are both non-negative. Therefore, their product $a_{ij}b_{ij}x_ix_j$ is also non-negative. Thus, $x^T(A \circ B)x$ is non-negative. Therefore, $A \circ B$ is positive semidefinite. □

The Loewner Partial Order

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Definition

The Loewner order is a partial order on the set of positive semidefinite symmetric matrices. For two positive semidefinite matrices A and B, we write $A \succcurlyeq B$ to denote that $A - B$ is positive semidefinite (and symmetric), and $A \succ B$ to denote that $A - B$ is positive definite.

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Theorem

Let $A, B \in \mathbb{M}_n$ be Hermitian and let $S \in \mathbb{M}_{n,m}$. Then

(a) *if $A \succ B$, then $S^*AS \succ S^*BS$,*

(b) *if $rank(S) = m$, then $A \succ B$ implies $S^*AS \succ S^*BS$.*

Proof.

Let $A, B \in \mathbb{M}_n$ be Hermitian and let $S \in \mathbb{M}_{n,m}$.

(a) Assume that $A \succ B$. Consider $S^*AS - S^*BS$.

We have, $S^*AS - S^*BS = S^*(A - B)S$

Since $A - B \succ 0$ and S^* is Hermitian.

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(b)Assume that $\text{rank}(S) = m$.

If $\text{rank}(S) = m$, let $A \succ B$.

consider $S^*AS - S^*BS$.

We have, $S^*AS - S^*BS = S^*(A-B)S$

Since $A - B \succ 0$ and S has full column rank.

Therefore $S^*AS \succ S^*BS$.

□

Conclusion

Positive definite and semidefinite matrices are essential mathematical constructs with diverse applications across different domains. They provide fundamental tools for understanding the geometry of vector spaces, ensuring the existence and uniqueness of solutions to problems, and optimizing various objective functions in both theoretical and practical settings.

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Thank You